

Systems of Linear Equations

Definition: Linear Equation

A **linear equation** is an equation in which each term is either a constant or the product of a constant and a single variable. The following are examples of linear equations:

$$2x + 3y = 5, \quad x - y + z = 4, \quad 6a - b + 2c = 0$$

Definition: Linear System

A **linear system** is a collection of linear equations using the same variables.

$$\begin{aligned}x + y + z &= 2 \\x - y + z &= 6 \\x + 2y - z &= -6\end{aligned}$$

A **solution** is the set of values that satisfies every equation simultaneously.

Definition: Gaussian Elimination

Gaussian Elimination solves a linear system by applying operations that simplify the system *without changing its solution set*, until the system is easy to solve directly. The three **elementary row operations** are:

- (1) **Swap** two rows: $R_i \leftrightarrow R_j$
- (2) **Scale** a row by a nonzero constant k : $R_i \rightarrow kR_i$
- (3) **Replace** a row with the sum of itself and a multiple of another row: $R_i \rightarrow R_i + kR_j$

Definition: Echelon Form

A system is in **echelon form** if:

1. Any rows consisting entirely of zeros are at the bottom.
2. In each nonzero row, the first nonzero entry (the **leading entry**, or **pivot**) lies strictly to the right of the leading entry in the row above it.

Visually, the pivots form a “staircase” pattern descending left to right. Once a system’s augmented matrix is in echelon form, its solution can be found using **back-substitution**.

$$\begin{aligned}2x + y - z &= 4 \\3y + 2z &= 1 \\5z &= 10\end{aligned}$$

Practice Problems

Solve each system below using Gaussian Elimination. Each system has exactly one solution.

1. $x + y = 5$
 $2x - y = 4$

2. $3x + 2y = 16$
 $x - y = 2$

3. $x + y + z = 6$
 $x - y + z = 2$
 $x + y - z = 0$

4. $2x + y - z = 3$
 $x - 3y + 2z = 10$
 $3x + y + z = 10$

5. $w + x + y + z = 5$
 $2w - x + y - z = -4$
 $w + 2x - y + 2z = 12$
 $3w - 2x + 4y + z = -2$

Example: Practice Problem 1

Original system:

$$x + y = 5$$

$$2x - y = 4$$

Apply $R_2 \rightarrow R_2 - 2R_1$ to reach echelon form:

$$x + y = 5$$

$$-3y = -6$$

Back-substitution gives $y = 2$, then $x = 3$.

$$x = 3, \quad y = 2$$

Example: Practice Problem 2

Original system:

$$3x + 2y = 16$$

$$x - y = 2$$

Apply $R_1 \leftrightarrow R_2$, then $R_2 \rightarrow R_2 - 3R_1$ to reach echelon form:

$$x - y = 2$$

$$5y = 10$$

Back-substitution gives $y = 2$, then $x = 4$.

$$x = 4, \quad y = 2$$

Example: Practice Problem 3

Original system:

$$x + y + z = 6$$

$$x - y + z = 2$$

$$x + y - z = 0$$

Apply $R_2 \rightarrow R_2 - R_1$ and $R_3 \rightarrow R_3 - R_1$ to reach echelon form:

$$x + y + z = 6$$

$$-2y = -4$$

$$-2z = -6$$

Back-substitution gives $y = 2$, $z = 3$, then $x = 1$.

$$x = 1, \quad y = 2, \quad z = 3$$

Example: Practice Problem 4

Original system:

$$2x + y - z = 3$$

$$x - 3y + 2z = 10$$

$$3x + y + z = 10$$

Apply $R_2 \rightarrow 2R_2 - R_1$ and $R_3 \rightarrow 2R_3 - 3R_1$, then $R_3 \rightarrow 7R_3 - R_2$ to reach echelon form:

$$2x + y - z = 3$$

$$-7y + 5z = 17$$

$$30z = 60$$

Back-substitution gives $z = 2$, $y = -1$, then $x = 3$.

$$\boxed{x = 3, \quad y = -1, \quad z = 2}$$

Example: Practice Problem 5

Original system:

$$w + x + y + z = 5$$

$$2w - x + y - z = -4$$

$$w + 2x - y + 2z = 12$$

$$3w - 2x + 4y + z = -2$$

Apply $R_2 \rightarrow R_2 - 2R_1$, $R_3 \rightarrow R_3 - R_1$, and $R_4 \rightarrow R_4 - 3R_1$ to eliminate w ; swap $R_2 \leftrightarrow R_3$; apply $R_3 \rightarrow R_3 + 3R_2$ and $R_4 \rightarrow R_4 + 5R_2$; then $R_4 \rightarrow 7R_4 - 9R_3$ to reach echelon form:

$$w + x + y + z = 5$$

$$x - 2y + z = 7$$

$$-7y = 7$$

$$21z = 63$$

Back-substitution gives $z = 3$, $y = -1$, $x = 2$, then $w = 1$.

$$\boxed{w = 1, \quad x = 2, \quad y = -1, \quad z = 3}$$