

Linear Systems: Many or No Solutions

Definition: The Three Possibilities

Every linear system has exactly one of three outcomes: **no solution** (the system is *inconsistent*), **exactly one solution** (covered in Lesson 1), or **infinitely many solutions** (the system is *dependent*). Gauss's method reveals which case we're in once the system reaches echelon form.

Definition: Inconsistent Systems (No Solution)

A system is **inconsistent** if no values satisfy every equation at once. In echelon form, this shows up as a row whose left side is entirely zero but whose right side is not:

$$0 = c, \quad c \neq 0$$

Once such a row appears, the system has no solution – there's nothing left to solve.

Definition: Dependent Systems (Infinitely Many Solutions)

A system is **dependent** if at least one equation carries no new information beyond the others. In echelon form, this shows up as a row of all zeros:

$$0 = 0$$

The variable that never becomes a pivot is called a **free variable** – it may take any value, and the remaining variables are written in terms of it.

Definition: General Solution

When a system has a free variable, we describe every solution at once using a parameter. Let the free variable equal a parameter t , then solve the pivot variables in terms of t . For example, a system with free variable z might have general solution:

$$x = 6 - 2t, \quad y = 1 + t, \quad z = t, \quad t \in \mathbb{R}$$

This is called the system's **general solution**.

Practice Problems

Solve each system below using Gauss's method. Some systems have no solution, some have infinitely many, and some have exactly one – determine which as you go.

1. $2x - y = 5$
 $x + y = 4$

2. $x + 3y = 5$
 $2x + 6y = 8$

3. $x - 2y = 3$
 $-3x + 6y = -9$

4. $x + y + z = 6$
 $2x + y - z = 3$
 $x + 2y + 4z = 11$

5. $x + y + z = 6$
 $2x + y - z = 3$
 $3x + 2y = 9$

Example: Practice Problem 1

Original system:

$$2x - y = 5$$

$$x + y = 4$$

Apply $R_2 \rightarrow 2R_2 - R_1$ to reach echelon form:

$$2x - y = 5$$

$$3y = 3$$

Back-substitution gives $y = 1$, then $x = 3$.

$$x = 3, \quad y = 1$$

Example: Practice Problem 2

Original system:

$$x + 3y = 5$$

$$2x + 6y = 8$$

Apply $R_2 \rightarrow R_2 - 2R_1$ to reach echelon form:

$$x + 3y = 5$$

$$0 = -2$$

The last row says $0 = -2$, a contradiction.

No solution

Example: Practice Problem 3

Original system:

$$x - 2y = 3$$

$$-3x + 6y = -9$$

Apply $R_2 \rightarrow R_2 + 3R_1$ to reach echelon form:

$$x - 2y = 3$$

$$0 = 0$$

The last row says $0 = 0$, so the system is dependent. Solving row 1 for x gives $x = 3 + 2y$. Let $y = t$:

$$x = 3 + 2t, \quad y = t, \quad t \in \mathbb{R}$$

Example: Practice Problem 4

Original system:

$$\begin{aligned}x + y + z &= 6 \\2x + y - z &= 3 \\x + 2y + 4z &= 11\end{aligned}$$

Apply $R_2 \rightarrow R_2 - 2R_1$ and $R_3 \rightarrow R_3 - R_1$ to eliminate x :

$$\begin{aligned}x + y + z &= 6 \\-y - 3z &= -9 \\y + 3z &= 5\end{aligned}$$

Apply $R_3 \rightarrow R_3 + R_2$ to reach echelon form:

$$\begin{aligned}x + y + z &= 6 \\-y - 3z &= -9 \\0 &= -4\end{aligned}$$

The last row says $0 = -4$, a contradiction.

No solution

Example: Practice Problem 5

Original system:

$$\begin{aligned}x + y + z &= 6 \\2x + y - z &= 3 \\3x + 2y &= 9\end{aligned}$$

Apply $R_2 \rightarrow R_2 - 2R_1$ and $R_3 \rightarrow R_3 - 3R_1$ to eliminate x :

$$\begin{aligned}x + y + z &= 6 \\-y - 3z &= -9 \\-y - 3z &= -9\end{aligned}$$

Apply $R_3 \rightarrow R_3 - R_2$ to reach echelon form:

$$\begin{aligned}x + y + z &= 6 \\-y - 3z &= -9 \\0 &= 0\end{aligned}$$

The last row says $0 = 0$, so the system is dependent, with z free. Row 2 gives $y = 9 - 3z$; substituting into row 1 gives $x = -3 + 2z$. Let $z = t$:

$x = -3 + 2t, \quad y = 9 - 3t, \quad z = t, \quad t \in \mathbb{R}$